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Some properties of zeros of polynomials with vanishing coefficients

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ABSTRACT

It is well known that when a polynomial whose coefficients are continuous functions of a parameter loses its degree then some of its zeros must vanish *at infinity*. In this paper, we consider such a situation: we examine how roots of a complex polynomial tend to *infinity* as some of its coefficients, including the leading one, tend to zero. We show, among other things, that in such a situation the unbounded paths traced by the roots of the polynomial have asymptotes; we also obtain their formulas. Some examples are presented to complete and illustrate the results.

1. Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\Re e(s)$, $\Im m(s)$, $|s|$ and $\bar{s}$ stand for the real part, the imaginary part, the moduli and the complex conjugate of a complex number s; i stands for the imaginary unit. The degree of a polynomial will be denoted by $\deg(\cdot)$.

A complex polynomial of degree n $(n \geqslant 1)$

$$f_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \qquad (1.1)$$
$$= a_n(s-s_1)\cdots(s-s_n), \quad a_n \neq 0$$

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is said to be *stable* if $\Re e(s_i) < 0$ $(i = 1, \ldots, n)$. A polynomial family P is said to be stable if $P \subset \mathscr{H}$, where $\mathscr{H}$ denotes the entire family of stable polynomials. $\overline{f_n}$ will denote the complex conjugate of polynomial (1.1), i.e.

$$\overline{f_n}(s) = \overline{a_n}s^n + \overline{a_{n-1}}s^{n-1} + \cdots + \overline{a_1}s + \overline{a_0}.$$

It is clear that if f is a (real or complex) polynomial of degree n, then $f\bar{f}$ is a real polynomial of degree $2n$ and $f \in \mathscr{H}$ if and only if $f\bar{f} \in \mathscr{H}$.

For a given polynomial family P, a set

$$\mathscr{R}(P) = \{s \in \mathbb{C} : \exists f \in P : f(s) = 0\}$$

is called *a root space* of P. Using this notion, one can express the stability of a set P as the inclusion $\mathscr{R}(P) \subset \{s \in \mathbb{C} : \Re e(s) < 0\}$.

Besides polynomial (1.1), we will also consider a complex polynomial of degree m $(1 \leqslant m \leqslant n)$

$$g_m(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0, \quad b_m \neq 0 \tag{1.2}$$

and a segment C_{f_n,g_m} generated by these polynomials, i.e.

$$C_{f_n,g_m} = \{\alpha f_n + (1-\alpha)g_m : \alpha \in [0, 1]\}.$$

At the end of this introductory section, note that all the discussions in this paper are of an elementary nature and hence are accessible to a wide audience of mathematicians.

2. A necessary condition for the stability of segments of polynomials

In this short section we focus our attention on the stability problem for segments of polynomials. Issues of this kind arise very frequently in many scientific fields, for example, in the control theory (where a classical transfer function has the form $H(s) = N(s)/D(s)$, where N and D are uncertain polynomials and the system described with the function H is stable if D is stable) or in the theory of dynamical systems (where one verifies the stability of solutions of continuous linear time-invariant systems whose coefficients are not known exactly) (see e.g. [1,2,3] and the references therein). The stability problem for polynomial families has been widely investigated in the past 25 years providing us with a very abundant literature. It was proven, among other things, that if a segment of polynomials generated by two *real* vertex polynomials, say f and g, with coefficients having the same signs, is stable then $|\deg f - \deg g| \leqslant 2$ (see [4]). Our goal now is to obtain a similar result applicable to polynomials with complex coefficients.

Lemma 1. *Consider two complex polynomials f_n and g_m of the form (1.1) and (1.2) and of degrees n and m $(n \geqslant m)$, respectively, and assume that*

$$\Re e(a_n\overline{b_m}) > 0. \tag{2.1}$$

The necessary condition for the stability of the segment C_{f_n,g_m} is that $n - m \leqslant 2$.

Proof. Suppose by contradiction that $n - m \geqslant 3$. Take any $f \in C_{f_n,g_m}$: $f = \alpha f_n + (1-\alpha)g_m$, for some $\alpha \in [0, 1]$, and consider the polynomial $f\bar{f}$. This polynomial may be written in the form

$$f(s)\bar{f}(s) = A_{2n}s^{2n} + A_{2n-1}s^{2n-1} + \cdots + A_1 s + A_0, \tag{2.2}$$

where the coefficients A_i $(i = 0, \ldots, 2n)$ are real-valued functions of a parameter $\alpha \in [0, 1]$ and of coefficients of the polynomials f_n and g_m. In particular

$$A_{n+m} = 2\alpha^2 \sum_{j=0}^{n-m} \Re e(a_{n-j}\overline{a_{m+j}}) + 2(1-\alpha)\alpha\Re e(a_n\overline{b_m}) \tag{2.3}$$

and

$$A_{n+m+l} = \alpha^2 \sum_{j=0}^{n-m-l} \Re e(a_{n-j}\overline{a_{m+l+j}}) \tag{2.4}$$

for $l = 1, \ldots, n-m$. Write the Hurwitz matrix associated with polynomial (2.2) (see [5]):

$$H(f\bar{f}) = \begin{pmatrix} A_{2n-1} & A_{2n} & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{2n-3} & A_{2n-2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & A_{2n-4} & A_{2n-3} & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & A_{n+m+2} & A_{n+m+3} & \cdots & \cdots \\ \cdots & \cdots & \cdots & A_{n+m} & A_{n+m+1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & A_1 & A_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & A_0 \end{pmatrix}$$

and consider its minor

$$M(\alpha) = \begin{vmatrix} A_{n+m+2} & A_{n+m+3} \\ A_{n+m} & A_{n+m+1} \end{vmatrix}.$$

It follows from (2.4) that the entries A_{n+m+1}, A_{n+m+2} and A_{n+m+3} do not depend on the coefficients $b_0, b_1, \ldots, b_m$. Moreover, they are non-zero as long as $n - m \geqslant 3$ and may be then expressed in the form

$$A_{n+m+1} = \alpha^2 h_1, \quad A_{n+m+2} = \alpha^2 h_2, \quad A_{n+m+3} = \alpha^2 h_3,$$

where the numbers h_1, h_2 and h_3, which can be obtained from (2.4), depend only on $a_0, \ldots, a_n$. Hence and from (2.3) we get

$$\begin{aligned} M(\alpha) &= \alpha^4 h_1 h_2 - \alpha^2 h_3 A_{n+m} \\ &= \alpha^4 \left[h_1 h_2 - 2h_3 \sum_{j=0}^{n-m} \Re e(a_{n-j}\overline{a_{m+j}}) - 2h_3 \left(\frac{1}{\alpha} - 1 \right) \Re e(a_n \overline{b_m}) \right]. \end{aligned} \tag{2.5}$$

On the other hand, the stability of the set C_{f_n, g_m} is equivalent to the stability of real polynomial (2.2) for every $\alpha \in [0, 1]$. Hence, for $\alpha \in (0, 1)$:

$$A_i > 0 \quad (i = 0, \ldots, 2n) \tag{2.6}$$

and

$$h_1 > 0, \quad h_2 > 0, \quad h_3 > 0. \tag{2.7}$$

Therefore, by combining assumption (2.1) with conditions (2.5) and (2.7), we get that for $\alpha \in (0, 1)$ sufficiently small the minor $M(\alpha)$ is negative. It is in contradiction with Asner's theorem saying that all the minors of the Hurwitz matrix associated with a real stable polynomial are non-negative (see [6]). This completes the proof. □

Note, that one can easily deduce from (2.3) that the segment generated by vertex polynomials (1.1) and (1.2) of different degrees (and only in that case) is stable only if $\Re e(a_n \overline{b_m}) \geqslant 0$. Assumption (2.1) of Lemma 1 is not therefore very restrictive.

3. Main results

In this section, we introduce some properties of zeros of a polynomial whose coefficients are continuous functions of a real parameter. We will consider a case in which the coefficients of a polynomial, including the leading one, vanish. One has to deal with such a problem, for example, when one considers the stability of segments of polynomials generated by vertex polynomials, say (1.1) and (1.2), of different degrees, say $n > m$. Lemma 1 proven in the previous section implies that in such a situation we may (and do) restrict ourselves to one of two following cases: $m = n - 1$ or $m = n - 2$, for $m \geqslant 1$.

3.1. Zeros of polynomials whose coefficients depend on a real parameter

It will be very convenient in the sequel of this section to use the following notation. For $t_0 > 0$, let

$$\omega(t_0) = (-t_0, t_0) \quad \text{or} \quad \omega(t_0) = (0, t_0) \quad \text{or} \quad \omega(t_0) = (-t_0, 0)$$

depending on a situation. Also, for a function $h : \omega(t_0) \to \mathbb{R}$ or $h : \omega(t_0) \to \mathbb{C}$ we will apply the convention:

- in case $\omega(t_0) = (-t_0, t_0)$ $\lim_{t\to 0^*} h(t)$ stands for $\lim_{t\to 0} h(t)$,
- in case $\omega(t_0) = 0, t_0)$ $\lim_{t\to 0^*} h(t)$ stands for $\lim_{t\to 0^+} h(t)$

and

- in case $\omega(t_0) = (-t_0, 0)$ $\lim_{t\to 0^*} h(t)$ stands for $\lim_{t\to 0^-} h(t)$.

The following theorem will play a key role in our further considerations (see [7]).

Theorem 2. *Consider a complex polynomial of the form*

$$\begin{aligned} g_m(s) &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \\ &= b_m (s - \alpha_1) \cdots (s - \alpha_m), \quad b_m \neq 0 \end{aligned}$$

and functions $a_k : \omega(t_0) \to \mathbb{C}$ $(k = 0, \ldots, n)$, *continuous in* $\omega(t_0)$ *and satisfying the following conditions*:

- $a_n(t) \neq 0$, *for* $t \neq 0$;
- $\lim_{t\to 0^*} a_k(t) = b_k$ $(k = 0, \ldots, m)$;
- $\lim_{t\to 0^*} a_k(t) = 0$ $(k = m + 1, \ldots, n)$.

Then zeros of the polynomial

$$\begin{aligned} f_n(s; t) &= a_n(t) s^n + a_{n-1}(t) s^{n-1} + \cdots + a_1(t) s + a_0(t) \\ &= a_n(t)(s - \alpha_1(t)) \cdots (s - \alpha_n(t)) \end{aligned}$$

can be labeled in such a way that

- $\lim_{t\to 0^*} \alpha_k(t) = \alpha_k$ $(k = 1, \ldots, m)$,
- $\lim_{t\to 0^*} |\alpha_k(t)| = +\infty$ $(k = m + 1, \ldots, n)$.

We now prove some new properties of roots of a polynomial whose leading coefficient vanishes.

Theorem 3. *Consider a complex polynomial of the form*

$$\begin{aligned} g_{n-1}(s) &= b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots + b_1 s + b_0 \\ &= b_{n-1}(s - \alpha_1) \cdots (s - \alpha_{n-1}), \quad b_{n-1} \neq 0 \end{aligned} \tag{3.1}$$

and functions $a_k : \omega(t_0) \to \mathbb{C}(k = 0, \ldots, n)$, *continuous in* $\omega(t_0)$ *and satisfying the following conditions*:

- $a_n(t) \neq 0$, *for* $t \neq 0$;
- $\lim_{t\to 0^*} a_k(t) = b_k \ (k = 0, \ldots, n-1)$;
- $\lim_{t\to 0^*} a_n(t) = 0$.

Then zeros of the polynomial

$$\begin{aligned} f_n(s;t) &= a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_1(t)s + a_0(t) \\ &= a_n(t)(s - \alpha_1(t)) \cdots (s - \alpha_n(t)) \end{aligned}$$

can be labeled in such a way that

(i) $\lim_{t\to 0^*} \alpha_k(t) = \alpha_k (k = 1, \ldots, n-1)$;

(ii) $\lim_{t\to 0^*} \Re e(\alpha_n(t)) = \begin{cases} -\infty & \textit{if } \lim_{t\to 0^*} \Re e\left(\frac{a_{n-1}(t)}{a_n(t)}\right) = +\infty \\ +\infty & \textit{if } \lim_{t\to 0^*} \Re e\left(\frac{a_{n-1}(t)}{a_n(t)}\right) = -\infty \\ -\Re e\left(d - \frac{b_{n-2}}{b_{n-1}}\right) & \textit{if } \lim_{t\to 0^*} \Re e\left(\frac{a_{n-1}(t)}{a_n(t)}\right) = d; \end{cases}$

(iii) $\lim_{t\to 0^*} \Im m(\alpha_n(t)) = \begin{cases} -\infty & \textit{if } \lim_{t\to 0^*} \Im m\left(\frac{a_{n-1}(t)}{a_n(t)}\right) = +\infty \\ +\infty & \textit{if } \lim_{t\to 0^*} \Im m\left(\frac{a_{n-1}(t)}{a_n(t)}\right) = -\infty \\ -\Im m\left(d - \frac{b_{n-2}}{b_{n-1}}\right) & \textit{if } \lim_{t\to 0^*} \Im m\left(\frac{a_{n-1}(t)}{a_n(t)}\right) = d. \end{cases}$

Proof. Condition (i) follows from Theorem 1. From the following Vieta's formula

$$\alpha_1(t) + \cdots + \alpha_n(t) = -\frac{a_{n-1}(t)}{a_n(t)} \quad \text{for any } t \neq 0,$$

we get equalities

$$\Re e(\alpha_n(t)) = -\Re e\left(\frac{a_{n-1}(t)}{a_n(t)} + \alpha_1(t) + \cdots + \alpha_{n-1}(t)\right)$$

and

$$\Im m(\alpha_n(t)) = -\Im m\left(\frac{a_{n-1}(t)}{a_n(t)} + \alpha_1(t) + \cdots + \alpha_{n-1}(t)\right)$$

from which we can easily derive conditions (ii) and (iii). □

From Theorem 3 we obtain the following:

Conclusion 1. Let f_n and g_{n-1} be two polynomials of the form (1.1) and (3.1), respectively, and consider, for $t \in \mathbb{R}$, the polynomial

$$\begin{aligned} tf_n(s) + g_{n-1}(s) &= ta_ns^n + (ta_{n-1} + b_{n-1})s^{n-1} + \cdots + (ta_0 + b_0) \\ &= ta_n(s - \beta_1(t)) \cdots (s - \beta_n(t)). \end{aligned} \tag{3.2}$$

Then zeros of polynomial (3.2) can be labeled in such a way that:

(i) $\lim_{t\to 0} \beta_k(t) = \alpha_k \ (k = 1, \ldots, n-1)$;

(ii) $\lim_{t\to 0^+} \Re e(\beta_n(t)) = \begin{cases} -\infty & \text{if } \Re e(\overline{a_n}b_{n-1}) > 0 \\ +\infty & \text{if } \Re e(\overline{a_n}b_{n-1}) < 0 \end{cases}$;

(iii) $\lim_{t\to 0^-} \Re e(\beta_n(t)) = \begin{cases} +\infty & \text{if } \Re e(\overline{a_n}b_{n-1}) > 0 \\ -\infty & \text{if } \Re e(\overline{a_n}b_{n-1}) < 0 \end{cases}$;

(iv) $\lim_{t\to 0} \Re e(\beta_n(t)) = -\Re e\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right)$, if $\Re e(\overline{a_n}b_{n-1}) = 0$;

(v) $\lim_{t\to 0^+} \Im m(\beta_n(t)) = \begin{cases} -\infty & \text{if } \Im m(\overline{a_n} b_{n-1}) > 0 \\ +\infty & \text{if } \Im m(\overline{a_n} b_{n-1}) < 0 \end{cases};$

(vi) $\lim_{t\to 0^-} \Im m(\beta_n(t)) = \begin{cases} +\infty & \text{if } \Im m(\overline{a_n} b_{n-1}) > 0 \\ -\infty & \text{if } \Im m(\overline{a_n} b_{n-1}) < 0 \end{cases};$

(vii) $\lim_{t\to 0} \Im m(\beta_n(t)) = -\Im m\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right)$, if $\Im m(\overline{a_n} b_{n-1}) = 0$.

From the above results one can easily deduce that if two vertex polynomials of the form (1.1) and (3.1) fulfil either of the conditions: $\Re e(\overline{a_n} b_{n-1}) = 0$ or $\Im m(\overline{a_n} b_{n-1}) = 0$, then the unbounded path traced by the roots of the segment appointed by these polynomials has an asymptote. On the other hand, it is worth pointing out that this unbounded path always has an asymptote; even if neither of that conditions holds (see Example 2).

We now prove the following theorem.

Theorem 4. *Let g_{n-2} be a real polynomial of the form*

$$\begin{aligned} g_{n-2}(s) &= b_{n-2}s^{n-2} + b_{n-3}s^{n-3} + \cdots + b_1 s + b_0 \\ &= b_{n-2}(s-\beta_1)\cdots(s-\beta_{n-2}), \quad b_{n-2} \neq 0. \end{aligned}$$

Also, let $a_k : \omega(t_0) \to \mathbb{R}$ $(k = 0, \ldots, n)$ be continuous functions such that

- $a_n(t) \neq 0$, *for* $t \neq 0$;
- $\lim_{t\to 0^*} a_k(t) = b_k$ $(k = 0, \ldots, n-2)$;
- $\lim_{t\to 0^*} a_{n-1}(t) = \lim_{t\to 0^*} a_n(t) = 0$;
- $\lim_{t\to 0^*} \frac{a_{n-1}(t)}{a_n(t)} = c$, *for some* $c \in \mathbb{R}$.

Then there exists a real number $t_1 > 0$ and zeros of the polynomial

$$\begin{aligned} p_n(s;t) &= a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_1(t)s + a_0(t) \\ &= a_n(t)(s-\beta_1(t))\cdots(s-\beta_n(t)) \end{aligned} \tag{3.3}$$

can be labeled in such a way that

(i) $\lim_{t\to 0^*} \beta_k(t) = \beta_k$ $(k = 1, \ldots, n-2)$;
(ii) $\lim_{t\to 0^*} |\beta_{n-1}(t)| = \lim_{t\to 0^*} |\beta_n(t)| = +\infty$;
(iii) $\beta_{n-1}(t) = \overline{\beta_n(t)}$ *in* $\omega(t_1)$;
(iv) $\lim_{t\to 0^*}(\beta_{n-1}(t) + \beta_n(t)) = \lim_{t\to 0^*} 2\Re e(\beta_{n-1}(t)) = \lim_{t\to 0^*} 2\Re e(\beta_n(t)) = \frac{b_{n-3}}{b_{n-2}} - c$;
(v) $\lim_{t\to 0^*} |\Im m(\beta_n(t))| = +\infty$.

Proof. Conditions (i) and (ii) follow from Theorem 2. To prove condition (iii) we argue by contradiction. Assume that for every $t_1 > 0$ there exists $t \in \omega(t_1)$ such that $\beta_{n-1}(t) \neq \overline{\beta_n(t)}$. Hence and from (ii) we get that at least three zeros of the polynomial $p_n(\cdot;t)$, i.e. $\beta_{n-1}(t)$, $\overline{\beta_{n-1}(t)}$ and $\beta_n(t)$, tend to infinity. By Theorem 2, it is impossible.

From condition (iii) and from Vieta's formulas we obtain that

$$\begin{aligned} \lim_{t\to 0^*}(\beta_{n-1}(t) + \beta_n(t)) &= \lim_{t\to 0^*}(\beta_{n-1}(t) + \overline{\beta_{n-1}(t)}) \\ &= \lim_{t\to 0^*} 2\Re e(\beta_{n-1}(t)) \\ &= \lim_{t\to 0^*} 2\Re e(\beta_n(t)) = \frac{b_{n-3}}{b_{n-2}} - c. \end{aligned}$$

This proves (iv). Condition (v) follows from conditions (iv) and (ii). $\square$

From the above theorem we get the following conclusion.

Conclusion 2. Consider two real polynomials

$$f_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad a_n \neq 0;$$
$$g_{n-2}(s) = b_{n-2}s^{n-2} + b_{n-3}s^{n-3} + \cdots + b_1 s + b_0 = b_{n-2}(s-\beta_1)\cdots(s-\beta_{n-2}), \quad b_{n-2} \neq 0.$$

Then there exists $t_1 > 0$ and zeros of the polynomial

$$tf_n(s) + g_{n-1}(s) = ta_n s^n + ta_{n-1}s^{n-1} + (ta_{n-2} + b_{n-2})s^{n-2} + \cdots + (ta_0 + b_0) = ta_n(s - \beta_1(t))\cdots(s-\beta_n(t))$$

can be labeled in such a way that

(i) $\lim_{t\to 0^*} \beta_k(t) = \beta_k$ $(k = 1,\dots,n-2)$,
(ii) $\lim_{t\to 0^*} |\beta_{n-1}(t)| = \lim_{t\to 0^*} |\beta_n(t)| = +\infty$,
(iii) $\beta_{n-1}(t) = \overline{\beta_n(t)}$ in $\omega(t_1)$,
(iv) $\lim_{t\to 0^*}(\beta_{n-1}(t) + \beta_n(t)) = 2\lim_{t\to 0^*} \Re e(\beta_n(t)) = \frac{b_{n-3}}{b_{n-2}} - \frac{a_{n-1}}{a_n}$,
(v) $\lim_{t\to 0^*} |\Im m(\beta_n(t))| = +\infty$.

3.2. An upper bound of the real parts of zeros of a segment of polynomials

In this part of the paper we will show that for some classes of polynomials the real parts of zeros of segments generated by these polynomials are bounded from above. Results of this kind are very useful, for example, when one examines the stability of segments consisting of polynomials of different degrees.

Theorem 5. *Consider two polynomials*

$$f_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad a_n \neq 0, \qquad g_m(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0, \quad b_m \neq 0 \tag{3.4}$$

that are stable and for which $\Re e(a_n\overline{b_m}) \geqslant 0$. *Then if either of the below conditions holds*

(i) $m = n-1 > 0$ *and polynomials* (3.4) *are complex*,
(ii) $m = n-2 > 0$ *and polynomials* (3.4) *are real*,

then the real parts of zeros of the segment C_{f_n,g_m} *are bounded from above*.

Moreover, if $\Re e(a_n\overline{b_m}) < 0$ *and* $n \neq m$, *then the real parts of zeros of the segment* C_{f_n,g_m} *are not bounded from above*.

Proof. Every element of the segment C_{f_n,g_m} is a polynomial of the form $h(s;\alpha) = \alpha f_n(s) + (1-\alpha)g_m(s)$, for some $\alpha \in [0,1]$. Moreover, for $\alpha \in (0,1)$ the degree of the polynomial $h(\cdot;\alpha)$ equals n; this polynomial can, therefore, be rewritten as

$$h(s;\alpha) = \alpha f_n(s) + (1-\alpha)g_m(s) = \alpha a_n(s - s_1(\alpha)\cdots(s - s_n(\alpha)))$$

or, equivalently, in the form:

$$h(s;\alpha) = \alpha f_n(s) + (1-\alpha)g_m(s) = (1-\alpha)(tf_n(s) + g_m(s)) = (1-\alpha)ta_n(s - x_1(t))\cdots(s-x_n(t)), \tag{3.5}$$

where $t = \frac{\alpha}{1-\alpha}$, $t \in (0, +\infty)$. It is now clear that for any fixed number $\alpha_0 \in (0, 1]$ the zeros $s_i(\alpha)$ $(i = 1, \ldots, n)$ are continuous functions of α, for $\alpha \in [\alpha_0, 1]$. It means that

$$\max_{\alpha \in [\alpha_0, 1]} \left(\max_{1 \leqslant i \leqslant n} \Re e(s_i(\alpha)) \right)$$

exists and hence, for any fixed $t_0 > 0$,

$$\max_{t \in [t_0, +\infty)} \left(\max_{1 \leqslant i \leqslant n} \Re e(x_i(t)) \right)$$

exists too. Hence, to prove the thesis it suffices to show that for some positive number t_0 and for every $t \in (0, t_0]$ the real parts of the functions $x_i(t)$ $(i = 1, \ldots, n)$ are bounded from above.
Let us consider now case (i). It follows from Conclusion 1 that zeros of polynomial (3.5) can be labeled in such a way that

- $\lim_{t \to 0^+} x_k(t) = \alpha_k$ $(k = 1, \ldots, n-1)$,
- $\lim_{t \to 0^+} \Re e(x_n(t)) = \begin{cases} -\infty, & \text{if } \Re e(\overline{a_n} b_{n-1}) > 0 \\ -\Re e\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right), & \text{if } \Re e(\overline{a_n} b_{n-1}) = 0 \end{cases}.$

It means that the roots $x_i(t)$ $(i = 1, \ldots, n)$ are bounded from above for every $t \in (0, t_0]$, where t_0 is any fixed positive number. This ends the proof in case (i).
Case (ii) If the polynomials f_n and g_{n-2} are real then from Conclusion 2 we get that roots of polynomial (3.5) can be labeled in such a way that

$$\lim_{t \to 0^+} \Re e(x_{n-1}(t)) = \lim_{t \to 0^+} \Re e(x_n(t)) = \frac{1}{2}\left(\frac{b_{n-3}}{b_{n-2}} - \frac{a_{n-1}}{a_n}\right).$$

From this, from point (i) of Conclusion 2 and from the continuity of the functions $x_i(t)$ $(i = 1, \ldots, n)$ for $t > 0$, we deduce that the real parts $\Re e(x_i(t))(i = 1, \ldots, n)$ are bounded from above.

To complete the proof recall that if $\Re e(a_n \overline{b_m}) < 0$ and $n \neq m$ then the segment C_{f_n, g_m} is not stable, and hence the real parts of its zeros cannot be bounded from above. The theorem is thus proved. $\square$

4. Polynomials stablewise extendable

A stable polynomial

$$\begin{aligned} g_m(s) &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \\ &= b_m (s - \beta_1) \cdots (s - \beta_m), \quad b_m \neq 0 \end{aligned} \tag{4.1}$$

is said to be *stablewise extendable* if for a given integer $n > m$ there exist (real if polynomial (4.1) is real and complex otherwise) numbers $a_{m+1}, \ldots, a_n$ $(a_n \neq 0)$ such that the polynomial

$$f_n(s) = a_n s^n + \cdots + a_{m+1} s^{m+1} + g_m(s)$$

is also stable.

Stablewise extendable polynomials play a very important role in many practical applications (see e.g. [3,8,9] for more details).

Now, we give a theorem that stands, among other things, that every complex and stable polynomial is stablewise extendable to a polynomial of arbitrarily high degree.

Theorem 6. *Let g_m be a complex polynomial of the form (4.1). Then, for a given integer $n > m$, for given numbers $\gamma_i < 0$ $(i = m+1, \ldots, n)$ and for any $\varepsilon > 0$ there exist complex numbers $a_{m+1}, \ldots, a_n$ $(a_n \neq 0)$ and zeros of the polynomial*

$$\begin{aligned} f_n(s) &= a_n s^n + \cdots + a_{m+1} s^{m+1} + g_m(s) \\ &= a_n (s - \alpha_1) \cdots (s - \alpha_n) \end{aligned}$$

can be labeled in such a way that

(i) $|\alpha_i - \beta_i| < \varepsilon (i = 1, \ldots, m)$,
(ii) $\Re e(\alpha_i) < \gamma_i \, (i = m+1, \ldots, n)$.

Proof. We prove the theorem by induction on $k = n - m$. First, we show that the thesis is true for $k = 1$. To do this, take any complex number c_{m+1}, but such that $\Re e(c_{m+1}\overline{b_m}) > 0$. Then, the polynomials $g_m(s)$ and $f_{m+1}(s) = c_{m+1}s^{m+1}$ satisfy the assumptions of Conclusion 1. For $t > 0$, let:

$$\begin{aligned} tf_{m+1}(s) + g_m(s) &= tc_{m+1}s^{m+1} + g_m(s) \\ &= tc_{m+1}(s - \widehat{\beta}_1(t)) \cdots (s - \widehat{\beta}_{m+1}(t)). \end{aligned}$$

It follows from Conclusion 1 that roots of the above polynomial can be labeled in such a way that

- $\lim_{t \to 0^+} \widehat{\beta}_i(t) = \alpha_i \; (i = 1, \ldots, m)$,
- $\lim_{t \to 0^+} \Re e(\widehat{\beta}_{m+1}(t)) = -\infty$.

Hence, for given $\varepsilon_1 > 0$ and $\gamma_{m+1} < 0$ there exists $t_0 > 0$ such that

- $|\widehat{\beta}_i(t_0) - \alpha_i| < \varepsilon_1 \; (i = 1, \ldots, m)$,
- $\Re e(\widehat{\beta}_{m+1}(t_0)) < \gamma_{m+1}$.

Putting $a_{m+1} = t_0 c_{m+1}$ we get the polynomial $a_{m+1}s^{m+1} + g_m(s)$ satisfying the desired conditions.

Assume now that the thesis is true for $n = m + k \; (k > 1)$. It implies that for any $\tilde{\varepsilon} > 0$ and $\tilde{\gamma}_i < 0 \; (i = m+1, \ldots, m+k)$ there exist complex numbers $\tilde{a}_{m+1}, \ldots, \tilde{a}_{m+k}$ $(\tilde{a}_{m+k} \neq 0)$ and zeros of the polynomial

$$\begin{aligned} g_{m+k}(s) &= \tilde{a}_{m+k}s^{m+k} + \cdots + \tilde{a}_{m+1}s^{m+1} + g_m(s) \\ &= \tilde{a}_{m+k}(s - \tilde{x}_1) \cdots (s - \tilde{x}_{m+k}) \end{aligned}$$

can be labeled in such a way that

$$\begin{aligned} &|\tilde{x}_i - \alpha_i| < \tilde{\varepsilon} \quad (i = 1, \ldots, m); \\ &\Re e(\tilde{x}_i) < \tilde{\gamma}_i \quad (i = m+1, \ldots, m+k). \end{aligned} \tag{4.2}$$

Our aim now is to prove that for any $\varepsilon > 0$ and $\gamma_i < 0 \; (i = m+1, \ldots, m+k+1)$ there exist complex numbers $a_{m+1}, \ldots, a_{m+k+1}$ $(a_{m+k+1} \neq 0)$ and zeros of the polynomial

$$\begin{aligned} g_{m+k+1}(s) &= a_{m+k+1}s^{m+k+1} + \cdots + a_{m+1}s^{m+1} + g_m(s) \\ &= a_{m+k+1}(s - x_1) \cdots (s - x_{m+k+1}) \end{aligned}$$

can be labeled in such a way that

$$\begin{aligned} &|x_i - \alpha_i| < \varepsilon \quad (i = 1, \ldots, m); \\ &\Re e(x_i) < \gamma_i \quad (i = m+1, \ldots, m+k+1). \end{aligned} \tag{4.3}$$

Consider now a complex number c_{m+k+1} for which $\Re e(\overline{c_{m+k+1}}\tilde{a}_{m+k}) > 0$. Then the polynomials $g_{m+k}(s)$ and $f_{m+k+1}(s) = c_{m+k+1}s^{m+k+1}$ satisfy the assumptions of Conclusion 1. For $t > 0$, let:

$$\begin{aligned} tf_{m+k+1}(s) + g_{m+k}(s) &= tc_{m+k+1}s^{m+k+1} + g_{m+k}(s) \\ &= tc_{m+k+1}(s - \beta_1(t)) \cdots (s - \beta_{m+k+1}(t)). \end{aligned}$$

It follows from Conclusion 1 that zeros of the above polynomial can be labeled in such a way that

- $\lim_{t\to 0^+} \beta_i(t) = \tilde{x}_i$ $(i = 1, \dots, m+k)$,
- $\lim_{t\to 0^+} \Re e(\beta_{m+k+1}(t)) = -\infty$.

It means that for a sufficiently small $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$|\beta_i(t_0) - \tilde{x}_i| < \frac{\varepsilon}{2} \quad (i = 1, \dots, m+k), \tag{4.4}$$
$$\Re e(\beta_{m+k+1}(t_0)) < \gamma_{m+k+1}.$$

Putting now in (4.2) $\tilde{\varepsilon} = \frac{\varepsilon}{2}$ and $\tilde{\gamma}_i = \gamma_i - \frac{\varepsilon}{2}$ $(i = m+1, \dots, m+k)$ and combining (4.2) and (4.4) we get the following inequalities:

$$|\beta_i(t_0) - \alpha_i| \leqslant |\beta_i(t_0) - \tilde{x}_i| + |\tilde{x}_i - \alpha_i| < \varepsilon \quad (i = 1, \dots, m) \tag{4.5}$$

and

$$\Re e(\beta_i(t_0)) < \Re e(\tilde{x}_i) + \frac{\varepsilon}{2} < \tilde{\gamma}_i + \frac{\varepsilon}{2} = \gamma_i \quad (i = m+1, \dots, m+k). \tag{4.6}$$

Putting $a_{m+k+1} = t_0 c_{m+k+1}$ we get the polynomial $a_{m+k+1}s^{m+k+1} + g_{m+k}(s)$ whose zeros satisfy inequalities (4.3). This completes the proof of the theorem. □

From the above theorem we can draw the following conclusion.

Conclusion 3. For any stable polynomial $g_m(s) = b_m s^m + \cdots + b_0 (b_m \neq 0)$ and for any integer $n > m$ and $\gamma_i < 0$ $(i = m+1, m+2 \dots, n)$ there exist numbers (real if the polynomial g_m is real and complex otherwise) $a_{m+1}, \dots, a_n$ $(a_n \neq 0)$ such that the polynomial

$$g_n(s) = a_n s^n + \cdots + a_{m+1}s^{m+1} + g_m(s)$$
$$= a_n(s - s_1)\cdots(s - s_n)$$

is stable and its zeros can be labeled in such a way that

$$\Re e(s_i) < \gamma_i \quad (i = m+1, m+2 \dots, n).$$

The results presented in this part of the paper are some generalizations of the main results of papers [8,9] in which similar problems, but concerning real polynomial families, were considered.

5. Examples

In this last part of the paper we show a few examples completing and illustrating the results presented in the previous sections.

Example 1. Consider two complex polynomials of the form

$$f_n(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2}, \quad a_n \neq 0,$$
$$g_{n-2}(s) = b_{n-2}s^{n-2}, \quad b_{n-2} \neq 0.$$

We can assume, without loss of generality, that $a_n = 1$. Then

$$tf_n(s) + g_{n-2}(s) = ts^n + a_{n-1}ts^{n-1} + (a_{n-2}t + b_{n-2})s^{n-2} \tag{5.1}$$
$$= s^{n-2}(ts^2 + a_{n-1}ts + (a_{n-2}t + b_{n-2})),$$

where $t > 0$. Using the notation introduced in Theorem 2, we have: $\alpha_1 = \cdots = \alpha_{n-2} = 0$ and hence, by Theorem 2, $n-2$ zeros of polynomial (5.1) tend to zero as t tends to zero, and the remaining zeros, i.e. $\alpha_{n-1}(t)$ and $\alpha_n(t)$, are solutions of the following quadratic equation with respect to s:

$$ts^2 + a_{n-1}ts + a_{n-2}t + b_{n-2} = 0. \tag{5.2}$$

Solutions of the above equation have the form

$$\alpha_{n-1}(t) = -\frac{1}{2}a_{n-1} + \frac{1}{2}\sqrt{|\Delta_1(t)|}\left(\cos\frac{\varphi(t)}{2} + i\sin\frac{\varphi(t)}{2}\right),$$
$$\alpha_n(t) = -\frac{1}{2}a_{n-1} - \frac{1}{2}\sqrt{|\Delta_1(t)|}\left(\cos\frac{\varphi(t)}{2} + i\sin\frac{\varphi(t)}{2}\right),$$

where

$$\begin{aligned}\Delta_1(t) &= a_{n-1}^2 - 4a_{n-1} - 4\frac{b_{n-2}}{t}\\ &= |\Delta_1(t)|(\cos\varphi(t) + i\sin\varphi(t)).\end{aligned}$$

Putting

$$a_{n-1}^2 - 4a_{n-1} = \lambda_1 + i\lambda_2 \quad \text{and} \quad -4b_{n-2} = \mu_1 + i\mu_2,$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$, we have

$$\Delta_1(t) = \left(\lambda_1 + \frac{\mu_1}{t}\right) + i\left(\lambda_2 + \frac{\mu_2}{t}\right).$$

Moreover, it follows from the assumptions that $\lim_{t\to 0^+}|\Delta_1(t)| = +\infty$.

Let us now consider five cases:

(i) $\mu_1 = 0$ and $\mu_2 \neq 0$. In such a case

$$\lim_{t\to 0^+}\varphi(t) = \frac{\pi}{2}\frac{\mu_2}{|\mu_2|}$$

and

$$\lim_{t\to 0^+}\Re e(\alpha_{n-1}(t)) = +\infty \quad \text{and} \lim_{t\to 0^+}\Re e(\alpha_n(t)) = -\infty.$$

(ii) $\mu_1 > 0$ and $\mu_2 = 0$. In such a case

$$\lim_{t\to 0^+}\varphi(t) = 0$$

and

$$\lim_{t\to 0^+}\Re e(\alpha_{n-1}(t)) = +\infty \quad \text{and} \quad \lim_{t\to 0^+}\Re e(\alpha_n(t)) = -\infty.$$

(iii) $\mu_1 < 0$ and $\mu_2 = 0$. In such a case

$$\lim_{t\to 0^+}\varphi(t) = \pi$$

and

$$\lim_{t\to 0^+}\Re e(\alpha_{n-1}(t)) = -\frac{1}{2}\Re e(a_{n-1}).$$

(iv) $\mu_1 > 0$ and $\mu_2 \neq 0$. In such a case

$$\lim_{t\to 0^+}\cos\varphi(t) = \frac{\mu_1}{\sqrt{\mu_1^2 + \mu_2^2}} > 0$$

and

$$\lim_{t\to 0^+}\Re e(\alpha_{n-1}(t)) = +\infty \quad \text{and} \quad \lim_{t\to 0^+}\Re e(\alpha_n(t)) = -\infty.$$

(v) $\mu_1 < 0$ and $\mu_2 \neq 0$. In such a case

$$\lim_{t\to 0^+} \cos\varphi(t) = \frac{\mu_1}{\sqrt{\mu_1^2+\mu_2^2}} < 0$$

and either

$$\lim_{t\to 0^+} \Re e(\alpha_{n-1}(t)) = +\infty \quad \text{and} \quad \lim_{t\to 0^+} \Re e(\alpha_n(t)) = -\infty$$

or

$$\lim_{t\to 0^+} \Re e(\alpha_{n-1}(t)) = -\infty \quad \text{and} \quad \lim_{t\to 0^+} \Re e(\alpha_n(t)) = +\infty.$$

Example 2. Let f_n and g_{n-1} be two complex polynomials of the form

$$f_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \quad a_n \neq 0,$$
$$g_{n-1}(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0, \quad b_{n-1} \neq 0,$$

whose leading coefficients satisfy the following conditions:

$$\Re e(\overline{a_n} b_{n-1}) \neq 0 \quad \text{and} \quad \Im m(\overline{a_n} b_{n-1}) \neq 0.$$

It follows from Conclusion 1 that there exists a continuous complex-valued function $\beta_n(t)$ of $t > 0$ such that

$$tf_n(\beta_n(t)) + g_{n-1}(\beta_n(t)) = 0, \quad \text{for } t > 0$$

and that

$$\lim_{t\to 0^+} |\Re e(\beta_n(t))| = +\infty \quad \text{and} \quad \lim_{t\to 0^+} |\Im m(\beta_n(t))| = +\infty.$$

We now show that in the case under consideration the path traced by $\beta_n(t)$ has an asymptote and also we give its formula.

To do this, it is convenient to introduce two polynomials defined as follows:

$$\tilde{f}_n(s;\varphi) = f_n(se^{-i\varphi}) \quad \text{and} \quad \tilde{g}_{n-1}(s;\varphi) = g_{n-1}(se^{-i\varphi}),$$

where $\varphi \in \mathbb{R}$ is a parameter. Writing these polynomials in their natural forms we have

$$\tilde{f}_n(s;\varphi) = \tilde{a}_n(\varphi)s^n + \tilde{a}_{n-1}(\varphi)s^{n-1} + \cdots + \tilde{a}_1(\varphi)s + \tilde{a}_0(\varphi),$$
$$\tilde{g}_{n-1}(s;\varphi) = \tilde{b}_{n-1}(\varphi)s^{n-1} + \tilde{b}_{n-2}(\varphi)s^{n-2} + \cdots + \tilde{b}_1(\varphi)s + \tilde{b}_0(\varphi),$$

where

$$\tilde{a}_k(\varphi) = a_k e^{-i\varphi k} \quad (k = 0, 1, \ldots, n),$$
$$\tilde{b}_k(\varphi) = b_k e^{-i\varphi k} \quad (k = 0, 1, \ldots, n-1).$$

Let now φ_0 be a solution of the below equation with respect to φ (it is not very difficult to show that the equation always has a solution):

$$\Re e(\overline{\tilde{a}_n(\varphi)}\tilde{b}_{n-1}(\varphi)) = 0. \tag{5.3}$$

Hence, after simple calculations, we obtain that one of the solutions is

$$\varphi_0 = \arctan\frac{\Re e(\overline{a_n} b_{n-1})}{\Im m(\overline{a_n} b_{n-1})}. \tag{5.4}$$

It follows from Conclusion 1, that the root space of the segment generated by vertex polynomials $\tilde{f}_n(\cdot;\varphi_0)$ and $\tilde{g}_{n-1}(\cdot;\varphi_0)$ has an asymptote. Moreover, by condition (iv) of Conclusion 1, in the complex plane $\{s = (x, y) : x = \Re e(s), y = \Im m(s)\}$, the asymptote may be written as

$$x = -\Re e\left(\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right)e^{i\varphi_0}\right). \tag{5.5}$$

On the other hand, it is clear that the set $\mathscr{R}(C_{f_n,g_{n-1}})$ can be obtained by rotating the root space of the segment generated by the vertex polynomials $\tilde{f}_n(\cdot;\varphi_0)$ and $\tilde{g}_{n-1}(\cdot;\varphi_0)$ by the angle $-\varphi_0$. Hence, after not very difficult calculations, we obtain that the straight line

$$\begin{aligned} y = {} & \frac{\Im m(\overline{a_n}b_{n-1})}{\Re e(\overline{a_n}b_{n-1})}\left(x + \Re e\left(\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right)e^{i\varphi_0}\right)\cos\varphi_0 + \sin\varphi_0\right) \\ & -\Re e\left(\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right)e^{i\varphi_0}\right)\sin\varphi_0 - \cos\varphi_0 \end{aligned} \tag{5.6}$$

is an asymptote of the set $\mathscr{R}(C_{f_n,g_{n-1}})$.

At the end, we give a numerical example.

Example 3. Consider a polytope (i.e. convex hull) generated by three complex polynomials of the form:

$$\begin{aligned} f_0(s) &= (1-2i)s^2 + (3-4i)s + 3 - 4i, \\ f_1(s) &= s^2 + (3-i)s + 3 - i, \\ f_2(s) &= s + 2 + i. \end{aligned}$$

It is obvious that the set $\mathscr{R}(C_{f_0f_1})$ is compact, while the sets $\mathscr{R}(C_{f_0f_2})$ and $\mathscr{R}(C_{f_1f_2})$ are not. It follows from Conclusion 1, that the set $\mathscr{R}(C_{f_1f_2})$ consists of roots, say $\gamma_1(\alpha)$ and $\gamma_2(\alpha)$ $(\alpha \in [0,1])$:

$$\alpha f_1(\gamma_i(\alpha)) + (1-\alpha)f_2(\gamma_i(\alpha)) = 0, \quad i = 1,2,$$

that satisfy the following conditions:

$$\lim_{\alpha\to 0^+}\gamma_1(\alpha) = -2 - i$$

and γ_2 tends to *infinity* in the following way:

$$\lim_{\alpha\to 0^+}\Re e(\gamma_2(\alpha)) = -\infty \quad \text{and} \quad \lim_{\alpha\to 0^+}\Im m(\gamma_2(\alpha)) = 2.$$

Similarly, the set $\mathscr{R}(C_{f_0f_2})$ consists of roots, say $\beta_1(\alpha)$ and $\beta_2(\alpha)$ $(\alpha \in [0,1])$, such that

$$\lim_{\alpha\to 0^+}\beta_1(\alpha) = -2 - i$$

and

$$\lim_{\alpha\to 0^+}\Re e(\beta_2(\alpha)) = -\infty \quad \text{and} \quad \lim_{\alpha\to 0^+}\Im m(\beta_2(\alpha)) = -\infty.$$

It follows from the previous example, that in this case the set $\mathscr{R}(C_{f_0f_2})$ has an asymptote determined by (5.6). Applying (5.4) and (5.5) to the polynomials f_0 and f_2 we have

$$\begin{aligned} \varphi_0 &= \arctan\frac{\Re e(\overline{a_n}b_{n-1})}{\Im m(\overline{a_n}b_{n-1})} = \arctan 0.5 \approx 0.46365, \\ x &= -\Re e\left(\left(\frac{a_{n-1}}{a_n} - \frac{b_{n-2}}{b_{n-1}}\right)e^{i\varphi_0}\right) \\ &= -\Re e\left(\left(\frac{3-4i}{1-2i} - 2 - i\right)e^{i0.46365}\right) \approx -0.44721 \end{aligned}$$

and hence we obtain the asymptote of the form: $y = 2x + 0.6$. Fig. 1, in which the root spaces of the segments $C_{f_0f_1}$, $C_{f_0f_2}$ and $C_{f_1f_2}$ and the asymptotes $y = 2x + 0.6$ and $y = 2$ (dashed lines) are presented, confirms the results graphically.

Consider now the segment generated by two polynomials, f_2 and f_α:

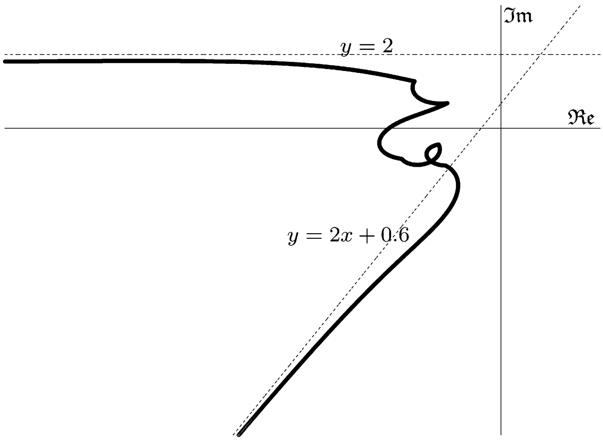

Fig. 1. Asymptotes (dashed lines) of the root space of all the edge polynomials from Example 3.

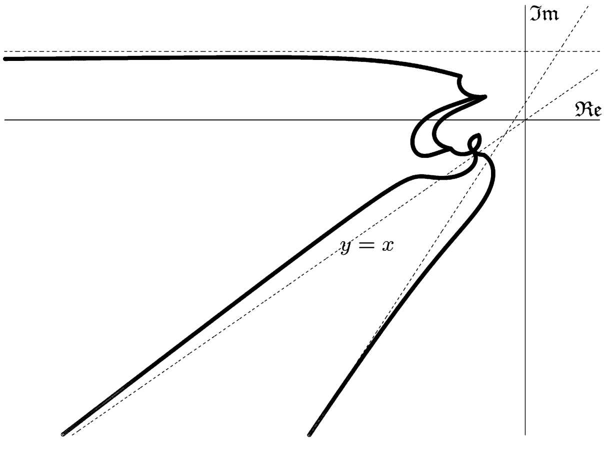

Fig. 2. The root spaces of all the edge polynomials and of the segment $C_{f_0 f_{1/2}}$ from Example 3 and their asymptotes (dashed lines).

$$f_\alpha(s) = (1-\alpha)f_0(s) + \alpha f_1(s),$$

where $\alpha \in (0, 1)$ is a parameter. For these polynomials, we have

$$\mathfrak{Re}(\overline{a_2}b_1) = \mathfrak{Re}\left(\overline{(1-\alpha)(1-2i)+\alpha}\right) = 1$$

and

$$\mathfrak{Im}(\overline{a_2}b_1) = \mathfrak{Im}\left(\overline{(1-\alpha)(1-2i)+\alpha}\right) = 2(1-\alpha),$$

where a_2 and b_1 stand for the leading coefficients of the polynomials f_α and f_2, respectively. It means, by Conclusion 1, that for every $\alpha \in (0, 1)$ the root space $\mathscr{R}(C_{f_2 f_\alpha})$ has an asymptote. Obviously, the asymptote has the form $y = A(\alpha)x + B(\alpha)$, where A and B are continuous functions of a parameter $\alpha \in (0, 1)$. In particular, a value of the coefficient A increases from 0 to 2, as α decreases from 1 to 0. For example, for $\alpha = \frac{1}{2}$ we get $\varphi_0 = \frac{1}{4}\pi$ and, by (5.6), the asymptote has the form $y = x$. Fig. 2 illustrates this situation.

Two last examples show how to determine asymptotes of a root space of a segment appointed by two vertex polynomials whose degrees differ by one. However, Example 3 gives us a bit more. To see this, recall the so-called *Edge Theorem*: for a polytope of real polynomials of the same degrees and for any simply connected domain $\Gamma \subset \mathbb{C}$, the root space generated by the entire polytope is contained in Γ if and only if all the root spaces generated by the edge polynomials are (see [10]). At present, we know at least a few very important generalizations of the Edge Theorem: the paper [11], for example, provides us with a version of the Edge Theorem that holds for complex polynomials of the same degrees and for domains whose complements are pathwise connected on the Riemann sphere (simple connectedness of a domain implies pathwise connectedness of its complement on the Riemann sphere; see [11]). It was also proven that for polytopes of *real* polynomials of different degrees and for domains satisfying the condition that each component of their complement is pathwise connected and contains at least a point on the real axis, the Edge Theorem also stays true (see [4]).

The main reason why we have presented Example 3 is to show that this last result does not hold in case of polytopes of complex polynomials. To see this, note that one can find a domain in the complex plane that fulfils the following conditions:

- it contains the root spaces of all the edge polynomials $C_{f_0f_1}$, $C_{f_0f_2}$, $C_{f_1f_2}$ from Example 3;
- it does not contain the root space of the entire polytope generated by the polynomials f_0,f_1,f_2;
- each component of its complement is pathwise connected and contains at least a point on the real axis.

As this domain one can take, for example, a sufficiently large ε-neighborhood of the root space generated by all the edge polynomials $C_{f_0f_1}$, $C_{f_0f_2}$ and $C_{f_1f_2}$ (see Fig. 1). It is obvious that such a domain fulfils the first and the third from the above conditions. In addition, it follows from Example 3, that the root space of the entire polytope is not included in the domain (it contains, among other things, the set $\mathscr{R}(C_{f_0f_{0.5}})$ that has a different asymptote than the root space of all the edge polynomials has, and hence it must leave its every ε-neighborhood; see Fig. 2). It means that the complex version of the Edge Theorem for polytopes of polynomials of different degrees and for domains satisfying the last from the above-mentioned conditions is not true.

6. Concluding remarks

In this paper, polynomials with coefficients depending on a real parameter, and as a special case – segments of polynomials, have been considered. It has been proven that when such polynomials drop the degree then the unbounded paths traced by their zeros have asymptotes; formulas for these asymptotes have been also obtained. Besides, it has been shown in Theorem 6 that every stable polynomial is *stablewise extendable* to a polynomial of arbitrarily high degree. Some examples completing and illustrating the results have been also given.

All the results presented in this note generalize some classical theorems concerning stable polynomials (quoted in the paper) or extend them to the complex case.

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